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# Compact symplectic manifolds of low cohomogeneity *

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**Abstract**

In this note we describe the equivariant diffeomorphism types of compact symplectic manifolds $M$ which admit a Hamiltonian action of a connected compact Lie group $G$ such that the quotient space $M/G$ has dimension 1. For a class of these manifolds we compute their small quantum cohomology algebra. We also construct some symplectic manifolds of cohomogeneity 2. © 1998 Elsevier Science B.V.

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## 1. Introduction

An action of a Lie group $G$ on a manifold $M$ is called of *cohomogeneity k* if the regular (principal) $G$-orbits have codimension $k$ in $M$. In other words the orbit space $M/G$ has dimension $k$. It is well known (see e.g. [Kir]) that homogeneous symplectic manifolds are locally symplectomorphic to coadjoint orbits of Lie groups whose symplectic geometry can be investigated in many aspects [Gr,HV,GK]. Our motivation is to find a wider class of symplectic manifolds via group approach, so that they could serve as test examples for many questions in symplectic geometry (and symplectic topology). In this note we describe all compact symplectic manifolds admitting a Hamiltonian action with cohomogeneity 1 of a connected compact Lie group. We always assume that the action is effective. We also remark that 4-manifolds admitting symplectic group actions (of cohomogeneity 1 or of $S^1$-action) have been studied intensively by many authors, see [Au] for references. In

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particular the classification of compact symplectic 4-manifolds admitting $SO(3)$-action of cohomogeneity 1 was done by Iglesias [I].

Let us recall that if an action of a Lie group $G$ on $(M, \omega)$ preserves the symplectic form $\omega$ then there is a Lie algebra homomorphism

$$g = Lie\, G \xrightarrow{\mathcal{F}_*} Vect_\omega(M), \tag{1.1}$$

where $Vect_\omega(M)$ denotes the Lie algebra of symplectic vector fields. The action of $G$ is said to be *almost Hamiltonian* if the image of $\mathcal{F}_*$ lies in the subalgebra $Vect_{\mathrm{Ham}}(M)$ of Hamiltonian vector fields. Finally, if the map $\mathcal{F}_*$ can be lifted to a homomorphism $g \xrightarrow{\mathcal{F}} C^\infty(M, \mathbb{R})$ (i.e. $\mathcal{F}_* v = sgrad\, \mathcal{F}_v$) then the action of $G$ is called *Hamiltonian*. In this note we shall prove the following theorem.

**Theorem 1.1.** *Suppose that a connected compact symplectic manifold $(M, \omega)$ is provided with a Hamiltonian action of a connected compact Lie group $G$ such that* $\dim M/G = 1$. *Then $M$ is $G$-diffeomorphic either to a $G$-invariant bundle over a coadjoint orbit of $G$ whose fiber is a complex projective manifold, or to a symplectic blow-down of such a bundle along two singular $G$-orbits.*

The main ingredients of our proof are the existence of the moment map, Duistermaat's–Heckman's theorem [DH] and the convexity theorem of Kirwan [Kiw]. For certain $G$-diffeomorphism types of these spaces we shall give a complete classification up to equivariant symplectomorphism (see Section 2).

In Section 3 we give a computation of the (small) quantum cohomology ring of some spaces admitting a Hamiltonian $U_n$-action with cohomogeneity 1 and discuss its corollaries.

We also consider the case of a symplectic action of cohomogeneity 2. In particular we get:

**Theorem 1.2.** *Suppose that a connected compact symplectic manifold $M$ is provided with a Hamiltonian action of a connected compact Lie group $G$ such that* $\dim M/G = 2$. *Then all the principal orbits of $G$ must be either (simultaneously) coisotropic or (simultaneously) symplectic. Thus a principal orbit of $G$ is either diffeomorphic to a $T^2$-bundle over a coadjoint orbit of $G$ (in the first case) or diffeomorphic to a coadjoint orbit of $G$ (in the second case).*

At the end of our note we collect in Appendix A some useful facts of the symplectic structures on the coadjoint orbits of compact Lie groups.

## 2. Classification of compact symplectic manifolds admitting a Hamiltonian action with cohomogeneity 1 of a connected compact Lie group

It is known [Br] that if an action of a compact Lie group $G$ on a connected compact oriented manifold $M$ has cohomogeneity 1 (i.e. $\dim M/G = 1$) then the topological space $Q = M/G = \pi(M)$ must be either diffeomorphic to the interval $[0, 1]$ or a circle $S^1$. The slice theorem gives us immediately that $G(m)$ is a principal orbit if and only if the image $\pi(G(m))$ in $Q$ is a interior point. In what follows we assume that $(M, \omega)$ is symplectic and the action of $G$ on $M$ is Hamiltonian. Under this assumption the quotient $Q$ is $[0,1]$ (see the proof below).

**Proposition 2.1.** *Let $G(m)$ be a principal orbit of a Hamiltonian $G$-action on $(M^{2n}, \omega)$. Then $G(m)$ is a $S^1$-bundle over a coadjoint orbit of $G$.*

*Proof.* In this case there exists a moment map

$$M^{2n} \xrightarrow{\mu} g^* : \langle \mu(m), w \rangle = \mathcal{F}_w(m). \tag{2.1}$$

For a vector $V \in T_*G(m)$ there is a vector $v \in g$ such that $V = \mathrm{d}/\mathrm{d}t_{t=0}(\exp tv) = sgrad\, \mathcal{F}_v$. Hence we get

$$\langle \mu_*(V), w \rangle = \mathrm{d}\mathcal{F}_w(V) = \{\mathcal{F}_w, \mathcal{F}_v\}(m) = \langle [w, v], \mu(m) \rangle, \tag{2.2}$$

which implies that $\mu$ is an equivariant map. Therefore the image $\mu(G(m))$ of any orbit $G(m)$ on $M$ is an adjoint orbit $G(\mu(m)) \subset g^*$.

To complete the proof of Proposition 2.1 we look at the preimage $\mu^{-1}\{\mu(m)\}$.

**Lemma 2.2.** *The preimage $\mu^{-1}\{\mu(m)\}$ is a closed submanifold of dimension at most 1 in $M$. If the preimage has dimension 1 then it is an orbit of a connected subgroup $S^1_m \subset G$.*

*Proof.* Clearly the preimage is a closed subset. We shall show that its dimension is at most 1. Let $V$ be a non-zero tangent vector to the preimage $\mu^{-1}\{\mu(m)\}$ at $x$. Then $\mu_*(V) = 0$. Using the formula

$$\langle \mu_*(V), w \rangle = \mathrm{d}\mathcal{F}_w(V) = \omega(sgrad\, \mathcal{F}_w, V) \tag{2.3}$$

for all $w \in g$ we conclude that $V$ is also a tangent vector to $G(m)$ and moreover $V$ annihilates the space $span\,\{\mathrm{d}\mathcal{F}_a \mid a \in g\}$ which has codimension 1 in $T^*M$. This proves the first statement. In particular there is an element $\bar{v} \in g$ such that $V = sgrad\, \mathcal{F}_{\bar{v}}$. Our claim on the manifold structure now follows from the fact that $\exp t\bar{v}(x) \subset \mu^{-1}\{\mu(m)\}$. This fact also yields the last statement on the orbit structure of the preimage. Finally, the preimage is connected because the quotient $\mu(G(m)) = G(m)/\{\mu^{-1}\}$ is simply connected (see Appendix A) and $G(m)$ is connected. □

*Proof of Proposition 2.1 (conclusion).* Now by dimension reason, if $m$ is a regular point of the $G$-action (i.e. $m$ is in a principal $G$-orbit) then $\mu^{-1}\{\mu(m)\}$ is a circle. This completes the proof of Proposition 2.1. □

Since $G$ is compact we can identify the coalgebra $g^*$ with $g$ via the Killing form. From the well-known convexity property of the moment map (see e.g. [Kiw]) we see that the quotient $Q = M/G$ is canonically diffeomorphic to the intersection of the image of the moment map $\mu(M)$ with a Weyl chamber $W$ in a Cartan subalgebra $Lie\, T \subset g$ (alternatively we can use Lemma 2.2 to get the diffeomorphism between two quotient spaces: $M/G$ and $\mu(M)/G$).

We shall improve Proposition 2.1 (see also Remark 2.15) in the following proposition, which is important for our construction of $G$-invariant symplectic structures. Let $m$ be a regular point of the $G$-action.

**Proposition 2.3.** *There is a Hamiltonian $S^1$-action on $M$ such that $G(\mu(m))$ is a symplectic quotient of $M$ under this $S^1$-action.*

*Proof.* It suffices to show that there is a Hamiltonian $S^1$-action on $M$ such that the orbit $G(m)$ coincides with the level set of a Hamiltonian function generating this $S^1$-action, and moreover the $S^1$-orbit through $m \in M$ coincides with the preimage $\mu^{-1}\{\mu(m)\}$. To construct a Hamiltonian function which generates this action we use the following simple lemma, a proof of it can be found in [McDS].

**Lemma 2.4.** *There is a compatible to $\omega$ almost complex structure $J$ on $M$ which is $G$-invariant.*

*Proof of Proposition 2.3 (continued).* Now let $H_G$ be a (unique up to constant and up to sign) $G$-invariant function on $M$ which satisfies the following condition:

$$2\pi \parallel grad\, H_G \parallel = L(\mu^{-1}\{\mu(m)\}), \tag{2.4}$$

where $L(\cdot)$ denotes the length for the metric defined by $\langle X, Y\rangle = \omega(X, JY)$. It is easy to see that $H_G$ generates the required Hamiltonian $S^1$-action. Clearly the symplectic form on $M$ descends to a symplectic form on the quotient $G(m)/S^1 = \mu(G(m))$. □

**Remark 2.5.**

(i) We also note that the stabilizer $St$ of this coadjoint orbit $\mu(G(m))$ is the product $G_m \cdot S^1_m$, where $G_m$ is the stabilizer of the orbit $G(m) \subset M$, and $S^1_m$ is a subgroup in $G$, generating the flows $\mu^{-1}\{\mu(m)\}$ (Lemma 2.2). More precisely, since $St$ is connected and $\dim S^1_m = 1$, $St$ is the "almost" direct product of the connected component $G^0_m$ of $G_m$ with $S^1_m$. Here "almost" means that on the level of Lie algebras the product is direct, and hence $G^0_m$ intersects with $S^1_m$ at a finite group $\mathbb{Z}^0_p$. Since $G_m \subset St$ we get that $G_m \cap S^1_m = \mathbb{Z}_p \supset \mathbb{Z}^0_p$.

(ii) The ("new") moment map $\mu_{S^1}$ determined by the $S^1$-action associated to $H_G$ also defines a diffeomorphism between $M/G$ and the image $\mu_G(M)$.

(iii) The Chern number of the $S^1$-fibration $G/G_m \to G/St$ is zero if and only if $G$ is an almost product of $S^1_m$ and a subgroup $G' \subset G$. In general, by Duistermaat–Heckman theorem this Chern number is defined uniquely by the intersection of $\mu(M)$ with a Weyl chamber.

We have seen that if an action of $G$ on $(M, \omega)$ is Hamiltonian with cohomogeneity 1 then the quotient space $M/G$ can be identified with the intersection of $\mu(M)$ with a Weyl chamber. Hence $G$ acts on the image $\mu(M)$ of $M$ with at most three orbit types: a regular one $G/Z(v)$ is the image of a regular orbit $G(m) \subset M$ and possibly other two orbit types $G/Z_{\min}$ and $G/Z_{\max}$ which are also coadjoint orbits of $G$. Henceforth we get:

**Lemma 2.6.** *There are only four possible cases:*
(I) $Z(v) \cong Z_{\min} \cong Z_{\max}$,
(II) $Z(v) \cong Z_{\min} \subset Z_{\max}$,
(III) $Z(v) \cong Z_{\max} \subset Z_{\min}$,
(IV) $Z_{\max} \supset Z(v) \subset Z_{\min}$.
*(Here "$\cong$" stands for conjugacy.)*

Now we shall describe $M$ according to four cases in Lemma 2.6.

Case (I): All symplectic quotients $G(m)/S^1$ are $G$-diffeomorphic. In this case by dimension reason and the fact that $G/Z(v)$ is simply connected, we see immediately that a singular orbit $G(m')$ is $G$-diffeomorphic to its image $\mu(G(m')) = G/Z(v)$. To specify the $G$-diffeomorphism type of $M$ it is useful to use the notion of segment [AA]. In our case we just consider the gradient flow of the function $H_G$ on $M$. After a completion and a reparametrization we get a geodesic segment $[s(t)]$, $t \in [0, 1]$, in $M$ such that the stabilizer of all the interior point $s(t)$, $t \in (0, 1)$, coincide with, say, $G_m$. (We observe that both $[s(t)]$ and the geodesic through $m$ with the initial vector $grad\ H_G(m)$ are characterized by the condition that every point in them is a fixed point of $G_m$.) Denote by $G_0$ and $G_1$ the stabilizers at singular points $s(0)$ and $s(1)$. Looking at the image of the gradient flow of $grad\ H_G$ under the moment map $\mu$ we conclude that $G_0 = G_1$.

**Proposition 2.7.** *In case* (I) *$M$ is $G$-diffeomorphic to $G \times_{G_0} S^2$, where $G_0 = (G_m^0 \times S_m^1)/\mathbb{Z}_p^0$ is the almost direct product of $G_m^0$ and $S_m^1$, and the left action of $G_0$ on $S^2$ is obtained via the composition of the projections $G_0 \to S_m^1/\mathbb{Z}_p^0$ with a Hamiltonian action of $S_m^1/\mathbb{Z}_p^0$ on $S^2$.*

*Proof.* First we identify the singular orbits in $M$ and in $G \times_{G_0} S^2$. The segment $[s(t)]$ extends this diffeomorphism to a diffeomorphism between $M$ and $G \times_{G_0} S^2$. Since $H_G$ is $G$-invariant it follows that this diffeomorphism is $G$-diffeomorphism. □

Now let us compute the cohomology ring $H^*(M, \mathbb{R})$ ( for $M$ in case (I)). Once we fix a Weyl chamber we get a canonical $G$-invariant projection $\Pi_\mu \colon \mu(M) \to \mu(G(m_0))$, where $G(m_0) = G/G_0$ is a singular orbit in $M$. Let $j := \Pi_\mu \circ \mu$ denote the projection $M \to B := \mu(G(m_0)) = G/Z(v) \cong G(m_0)$. Geometrically $j(x) = j(\mu^{-1}(\mu(x)))$ is the limit of the flow generated by $grad\ H_G$ passing through $x$. Note that $G(m_0)$ is the image of a section $s : B \to M$ of our $S^2$-bundle, and in what follows we shall identify the base $B$ with its section $G(m_0)$. Let $f$ denote the Poincare dual to the homology class $[G(m_0)] \in H_*(M, \mathbb{R})$.

Let $x_0 \in H^2(\mu(G(m_0)), \mathbb{R})$ be the image of the Chern class of the $S^1$-bundle $G(m) \to G(m_0)$, where $G(m)$ is a regular orbit $G/G_m$ (or in other words, $x_0$ is the Chern class of the normal bundle over $G(m_0)$ with the induced (almost) complex structure).

Let $\{x_i, R_l\}$ denote the set of generators and their relations in cohomology ring $H^*(\mu(G(m_0)), \mathbb{R})$ (see [Bo], correspondingly Proposition A.4 in Appendix A).

**Proposition 2.8.** *We have the following isomorphism of additive groups:*

$$H^*(G \times_{G_0} S^2, \mathbb{R}) = H^*(G/G_0, \mathbb{R}) \otimes H^*(S^2, \mathbb{R}). \tag{2.5}$$

*The only non-trivial relation in the algebra* $H^*(M, \mathbb{R})$ *are* (R1), (R2), *with*

$$f(f - j^*(x_0)) = 0. \tag{R2}$$

*Proof.* Statement (2.5) on the additive structure of $H^*(M, \mathbb{R})$ follows from the triviality of the cohomology spectral sequence of our $S^2$-bundle. Clearly (R1) remains the relation between the generators $\{j^*(x_i)\}$ in $H^*(M, \mathbb{R})$. To show that relation (R2) holds we have two arguments. One is in the proof of Lemma 2.13 and the other is here. Using the intersection formula for $x_0$ we notice that the restriction of $(f - j^*(x_0))$ to $G(m_0)$ is trivial. Thus to get relation (R2) it is enough to verify that the value of LHS of (R2) on the cycles in $M$ of the forms $j^{-1}([C])$ is always zero, where $[C] \in H_2(B, \mathbb{Z})$. Denote by $PD_M(\cdot)$ the Poincare dual in $M$. From the identity

$$(PD_M([G(m_0)]))^2 = PD_M([G(m_0) \cap G(m_0)]) = PD_M[PD_B(x_0)]$$

we get $f^2 = PD_M[PD_B(x_0)]$. Now it follows that

$$f^2(j^{-1}([C])) = f([C]). \tag{R2a}$$

On the other hand, since the restriction of the 2-form representing $j^*(x_0)$ to the fiber $S^2$ is vanished, we can apply the Fubini formula to the integration of a differential form representing the class $f \cdot j^*(x_0)$ (we can assume that $[C]$ is represented by a pseudo-manifold). In the result we get that

$$f \cdot j^*(x_0)(j^{-1}([C])) = x_0([C])) = f([C]). \tag{R2b}$$

Thus (R2) is a relation in $H^*(M, \mathbb{R})$. Finally the statement that (R2) is the only "new" relation in $H^*(M, \mathbb{R})$ follows from the triviality of our spectral sequence. □

**Remark 2.9.**

(i) If we take the other singular orbit $G(m_1) = G/G_1$ then the Chern class of the $S^1$-bundle : $G(m) \to G(m_1)$ is $-x_0$ (after an obvious identification $G(m_0)$ with $G(m_1)$ since $G(m_1)$ can be considered as another section (at infinity) of our $S^2$-bundle). It is also easy to see that the restriction of $f$ on $G(m_1)$ is zero since $G(m_0)$ has no common point with $G(m_1)$.

(ii) If $S^1_m$ is a normal subgroup of $G$ then $M$ is a direct product $G/G_0 \times S^2$. In this case any $G$-invariant metric on $M = G \times_{G_0} S^2$ can be recovered from a $G$-invariant metric on $G/G_0$ and an $S^1_m$-invariant metric on the fiber $S^2$. If $S^1_m$ is not a normal subgroup of $G$ then there is an element $g \in G$ such that $Ad_g(Lie\, S^1_m) \neq Lie\, S^1_m$. This implies that a $G$-invariant metric on $S^1$-fibration $G/G^0_m$ is defined uniquely by the quotient metric on $G/G_0$. In this case the space of $G$-invariant metrics on $M$ is one-to-one corresponding to the space of $G$-invariant metrics in the image of $M$ under the moment map $\mu$. Here we observe that the foliation $\mu^{-1}\{\mu(m)\}$ (and hence the quotient $M/S^1 = M/\mu^{-1}(\mu)$) can be defined intrinsically by the action of $G$ on $M$ (see Lemma 2.13).

**Proposition 2.10.** *Let $M^{2n}$ be in case* (1) *of Lemma* 2.6 *and let us keep the notation in Proposition* 2.8 *for $M$. Then $M^{2n}$ admits a $G$-invariant symplectic form $\omega$ in a class $[\omega] \in H^2(M^{2n}, \mathbb{R})$ if and only if $[\omega] = j^*(x) + \alpha \cdot f$ with $\alpha > 0$, and $(x + t \cdot \alpha \cdot x_0)^{n-1} > 0$ for all $t \in [0, 1]$. In particular $M^{2n}$ always admits a $G$-invariant symplectic structure such that the action of $G$ on $M$ is Hamiltonian.*

*Proof.* Let $[\omega] = j^*(x) + \alpha \cdot f$ with $x \in H^2(G/G_0, \mathbb{R})$. The condition that $\alpha > 0$ follows from the fact that the restriction of $\omega$ to each fiber $S^2$ is positive. (Here we assume that the orientation of $M$ agrees with that of $G(m)$ and the frame $(grad\ H_G, sgrad H_G)$. The last frame is a frame of tangent space to the fiber $S^2$.) Thus the "only if" statement now follows trivially from the Duistermaat–Heckman theorem.

Now let us assume that the class $[\omega]$ satisfies the condition in Proposition 2.10. Clearly all these cohomology classes $(x + t \cdot \alpha \cdot x_0)$, $0 \le t \le 1$, are realized by $G$-invariant symplectic forms by our condition (see also Remark A.5). We fix a 1-parameter family of $G$-invariant metrics on $G/G_0$ which are also compatible with these symplectic forms. According to Remark 2.9(ii) we can construct a $G$-invariant metric on $M$ which compatible with this family of $G$-invariant metrics on $G/G_0$. Lifting to $M$ we can define the restriction $\bar{\omega}$ of $\omega$ to each orbit $G(m)$. We normalize the $G$-invariant metric on $M$ in the direction $grad\ H_G$ orthogonal to the orbit $G(m)$ such that the following condition holds:

$$grad\ H_0(\bar{\omega})(m) = -L(\mu^{-1}\{\mu(m)\}) j^* x_0, \tag{2.6}$$

where $grad\ H_0 := grad\ H_G / \parallel grad\ H_G \parallel$ (we can normalize this metric by multiplying the length of $grad\ H_G$ with a positive function, because $\alpha > 0$). By the construction $\bar{\omega}$ is a $G$-invariant 2-form on $M$ whose rank is $(n-1)$. Denote by $\alpha \hat{f}_G$ the $G$-invariant 2-form on $M$ whose restriction to each fiber $S^2$ is compatible with the restriction of the $G$-invariant metric to $S^2$. We put $\omega = \bar{\omega} + \alpha \hat{f}_G$. By the construction $\omega$ is a $G$-invariant 2-form of maximal rank on $M$. We claim that $\omega$ is a symplectic form realizing the class $j^*(x) + \alpha \cdot f$. To verify the closedness of $\omega$ it suffices to establish the following identities:

$$\mathrm{d}\omega(sgrad\ H_0, grad\ H_0, V_1) = 0, \tag{2.7}$$

$$\mathrm{d}\omega(sgrad\ H_0, V_1, V_2) = 0, \tag{2.8}$$

$$\mathrm{d}\omega(grad\ H_0, V_1, V_2) = 0, \tag{2.9}$$

$$\mathrm{d}\omega(V_1, V_2, V_3) = 0, \tag{2.10}$$

for all $V_i$ in the normal bundle to the fiber $S^2$ and here $sgrad\ H_0$ denotes the unite vector in $\ker \bar{\omega}_{|G(m)}$, whose orientation agrees with that of the fiber $S^1$. Using the formula

$$\begin{aligned} 3\,\mathrm{d}\omega(X, Y, Z) = {}& X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y)) \\ & - \omega([X, Y], Z) - \omega([Y, Z], X) - \omega([Z, X], Y) \end{aligned} \tag{2.11}$$

we easily get that the LHS of (2.8) equals $\mathrm{d}\bar{\omega}_{|G(m)} = 0$.

Applying (2.11) to (2.10) we also get that $\mathrm{d}\omega(V_1, V_2, V_3) = \mathrm{d}\bar{\omega}(V_1, V_2, V_3) + \mathrm{d}\alpha \hat{f}_G (V_1, V_2, V_3) = 0 + 0 = 0$.

To compute (2.7) we assume that $V_i$ is generated by the action of a 1-parameter subgroup of $G$ (acting on $M$). Taking into account that $[sgrad\, H_0, grad\, H_0] \in \ker \bar{\omega}$ we get

$$-3\,\mathrm{d}\omega(sgrad\, H_0, grad\, H_0, V) = \alpha \hat{f}_G([grad\, H_0, V], sgrad\, H_0) - \alpha \hat{f}_G([sgrad\, H_0, V], grad\, H_0]). \tag{2.12}$$

RHS of (2.12) is zero since $\alpha \hat{f}_G$ is $G$-invariant. Hence (2.7) is zero.

To compute (2.9) we also assume that $V_i$ is generated by the action of a 1-parameter subgroup of $G$. Since $H_G$ is $G$-invariant we get $[V_i, grad\, H_G] = 0 = [V_i, grad\, H_0]$. Applying (2.11) to LHS of (2.9) we get

$$3\,\mathrm{d}\omega(grad\, H_0, V_1, V_2) = grad\, H_0(\bar{\omega}(V_1, V_2)) - \alpha \hat{f}_G([V_1, V_2], grad\, H_0). \tag{2.13}$$

By the choice of $\alpha \hat{f}_G$ the second term in RHS of (2.13) equals $-\langle sgrad\, H_0, [V_1, V_2]\rangle$.

Let us denote by $M^{\mathrm{reg}}$ the set of regular points of the $G$-action on $M$. By the choice of $V_i$ and $\bar{\omega}$ (see (2.6)) the first term in LHS of (2.13) equals $(1/2\pi)\,\mathrm{d}\theta(V_1, V_2) \cdot L(\mu^{-1}\{\mu(m)\})$ $= -(1/4\pi)\theta([V_1, V_2]) \cdot L(\mu^{-1}\{\mu(m)\})$, where $\theta$ is the connection form on the $S^1$-fibration $M^{\mathrm{reg}}$. In the presence of the (lifted) $S^1$-invariant metric on $M^{\mathrm{reg}}$ we can take $\theta([V_1, V_2])$ as $4\pi \langle sgrad\, H_0, [V_1, V_2]\rangle / L(\mu^{-1}\{\mu(m)\})$.

It follows that the LHS of (2.13) equals zero. This completes the proof of the closedness of $\omega$. Looking at the restriction of $\omega$ to $G(m_1)$ and $G(m_0)$ we conclude that $\omega$ represents the class $[j^*(x) + \alpha \cdot f]$.

The statement on the existence of a $G$-invariant symplectic structure follows from the fact that $G/G_0$ always admits a class $x$ such that $x^{n-1} > 0$. Since we can multiply $x$ with a large positive constant $\lambda$, the class $(x + tx_0)^{n-1}$ is also positive for all $t \in [0, 1]$ and we can apply the first statement here.

The vanishing of the first Betti-number of $M$ implies that the action of $G$ is almost Hamiltonian and hence Hamiltonian because $G$ is compact. □

*Cases* (II) *and* (III) (in Lemma 2.6): If we are interested in the $G$-diffeomorphism type then these cases are equivalent by changing the sign of the function $H_G$. Thus we shall consider case (II): $Z(v) = Z_{\min}$. (We can also consider case (I) as a subcase of case (II).) Since the two singular orbits are the critical level set for the function $H_G$ we get that at every singular point $m \in M$ the preimage $\mu^{-1}\{\mu(m)\}$ consists of exactly one point. Hence $G_{\max} = Z_{\max}$ and $G_{\min} = Z_{\min}$. Note that $G_{\max}/G_{\mathrm{reg}} = S^k$ by the slice theorem. On the other hand we have $Z(v) = G_{\mathrm{reg}} \times S^1$. Because $Z_{\max}/Z(v)$ is always of even dimension we have $Z_{\max}/Z(v) = \mathbb{C}P^{(k-1)/2} = \mathbb{C}P^l$.

**Lemma 2.11.** *In case* (II) *we have the following decompositions:* $G_{\max} = SU_{l+1} \times G_0$, $G_{\mathrm{reg}} = SU_l \times G_0$ *and* $Z_v = S(U_l \times U_1) \times G_0$, *where the inclusion* $SU_l \to S(U_l \times U_1) \to SU_{l+1}$ *is standard.*

*Proof.* By checking Table A.3 (in Appendix A) of possible coadjoint orbit types we see that the pair $(Z(v), Z_{\max} \cong G_{\max})$ in case (II) can be only:

Series (A): $Z_{\max} = S(U_{l+1} \times \cdots \times U_{n_k})$. Then $Z(v) = S(U_l \times S^1 \times \cdots \times U_{n_k})$ and $G_{\mathrm{reg}} = S(U_l \times \cdots \times U_{n_k})$.

Series (B), (D): $Z_{\max} = U_{l+1} \times \cdots \times SO_{2n_k+(1)}$, $Z(v) = U_l \times U_1 \times \cdots \times SO_{2n_k+(1)}$ and $G_{\mathrm{reg}} = U_n \times \cdots \times SO_{2h_k+(1)}$.

Series (C): Analogous to (B) and (D).

Exceptional case: The same (see Table A.3 in Appendix A).

If $G$ is a product of compact Lie groups then its coadjoint orbits are product of coadjoint orbits of each factors. It it well known that every compact group Lie admits a finite covering which is a product of a torus and compact simply connected Lie groups whose algebra are simple. Thus to prove Lemma 2.11 in general case it suffices to consider the above cases. □

The following proposition is an analog of Propositions 2.8 and 2.10.

**Proposition 2.12.** *Let $M$ be in case* (II). *Then $M$ is $G$-diffeomorphic to a $G$-invariant $\mathbb{C}P^{l+1}$-bundle over $G/G_{\max}$. There is a $G$-invariant symplectic structure on $M$ and the action of $G$ is Hamiltonian with respect to this structure.*

*Proof.* To prove the first statement we consider the projection $M \to G/G_{\max}: x \mapsto \mu(x) \mapsto \Pi(\mu(x))$, where $\Pi$ is a canonical projection from $\mu(M)$ to the singular coadjoint orbit $G/G_{\max}$. We recall that this canonical projection can be chosen by using the intersection of $\mu(M)$ with a Weyl chamber (see [Kir]). By Lemma 2.11 the fiber of this projection is the sum $D^{2(l+1)} \cup S^{2l+1} \times I \cup \mathbb{C}P^l$ and isomorphic to $\mathbb{C}P^{l+1}$. Clearly this fiber consists of all trajectories of the flows $grad\ H_G$ which end up at a point in the singular orbit $G/G_{\max}$. Hence the action of $G$ sends a fiber to a fiber.

It is also easy to describe the cohomology algebra of $M$ by the method in Proposition 2.8. Namely we denote by $f$ the Poincare dual to the singular orbit $G/G_{\min}$ of codimension 2 in $M$. Since the singular orbit $G/G_{\min}$ intersects the fiber $\mathbb{C}P^{l+1}$ at a hyperplane $\mathbb{C}P^l$, the restriction of $f$ on the fiber $\mathbb{C}P^{l+1}$ is the generator of the cohomology group $H^2(\mathbb{C}P^n, \mathbb{R})$. Henceforth the ring $H^*(M, \mathbb{R})$ is generated by $\{f, x_i\}$, where $x_i$ are the pull-back of the generators of the ring $H^*(G/G_{\max}, \mathbb{R})$ (compare (2.5)). Let (R1) denote the relation between $x_i$ in $H^*(G/G_{\max}, \mathbb{R})$, and let $P_{\min}$ denote the Poincare dual to the singular orbit $G/G_{\min} \subset M$. Put (R2) $= f \cdot P_{\min}$. It is easy to see (using the fact that two singular orbits have no common points and the associativity of the cap action) that (R1) and (R2) are the only relation in $H^*(M, \mathbb{R})$. (Now apply to the case in Proposition 2.8 we observe that $P_{\min} = f - x_0$.)

To show the existence of a $G$-invariant symplectic structure on $M$ we use the lifting construction of a family of invariant symplectic structures on $G/G_{\max}$ as in the proof of Proposition 2.10. Here the main observation is the following.

**Lemma 2.13.** *Let $G(m)$ be a principal orbit and $p_H$ denotes the projection from $M \setminus (G/G_{\max}) \to G/G_{\min}$ which is defined by the gradient flow of $H_G$. Then the characteristic leaf $\mu^{-1}\{\mu(m)\}$ coincides with $p_H^{-1}(m) \cap G(m)$.*

*Proof.* The projection of the gradient flow of $H_G$ is also a gradient flow of a $G$-invariant function $H$ on $\mu(M)$. The slice theorem tells us that along the gradient flow of $H$ all the stabilizer groups coincide. Hence follows statement. □

*Proof of Proposition 2.12 (continued).* Let $[\omega] = x + \alpha \cdot f$ be an element in $H^2(M, \mathbb{R})$. Clearly a necessary condition for the existence of a symplectic form $\omega$ in the class $[\omega]$ is that $x^l > 0$, $\alpha > 0$ and for all $t \in [0, 1)$ we have that the restriction of the cohomology class $(j^*x + t \cdot \alpha \cdot f)$ to the big orbit $G/G_{\min}$ is also symplectic. (That follows from the Duistermaat–Heckman theorem or Kirwan's theorem.) Here the restriction of $f$ to the big orbit $G/G_{\min}$ is the first Chern class of the $S^1$-fibration $G(m) \stackrel{p_H}{\to} G/G_{\min}$. Now let the class $[\omega] \in H^2(M, \mathbb{R})$ satisfy the above condition. Lifting the family of symplectic forms on the quotient $(M \setminus (G/G_{\max}))/S^1$ we get a symplectic form on $M \setminus (G/G_{\max})$ (see the proof of Proposition 2.10). By the construction the lifted form extends continuously and non-degenerately on the whole $M$ such that its restriction to the small orbit equals $j^*(x)$. The closedness is also automatically valid. Considering the restriction of the lifted form to the two singular orbits yields that our form realizes the cohomology class $j^*(x) + \alpha \cdot f$.

To show the existence of a $G$-invariant symplectic structure on $M$ we use the fact that $G_{\max}/G_{\min} = \mathbb{C}P^l$. Under this condition we can find a $G$-invariant 2-form $\bar{x}$ in a class $x \in H^2(G/G_{\max}, \mathbb{R})$ such that $\bar{x}$ is a $G$-invariant symplectic form and $j^*(\bar{x}) + t\bar{f}$ is a $G$-invariant symplectic form realizing the cohomology class $j^*(x) + t \cdot f$ for $t \in (0, 1]$. (Here we construct a $G$-invariant 2-form on $G/G_{\max}$ by $G$-invariant extension of a $G_{\max}$-invariant 2-form $\langle \alpha, [X, Y] \rangle$ in the $T_e(G/G_{\max})$.) □

*Case* (IV) (in Lemma 2.6): First we note that according to the theorem of Duistermaat–Heckman this case never happens when $\dim M = 4$, because the volume of a orbit $G(\mu(m))$ tends to zero when $\mu(m)$ tends to a point in a singular (coadjoint) orbit. The same argument as in case (II) shows us that $G_{\max} \cong Z_{\max}$, $G_{\min} \cong Z_{\min}$ and $Z_{\max}/Z(v) = \mathbb{C}P^l$, $Z_{\min}/Z(v) = \mathbb{C}P^k$.

**Proposition 2.14.** *Suppose that $M$ is in case* (IV). *Then $M$ is $G$-diffeomorphic to a $G$-invariant bundle over a coadjoint orbit of $G$ or to the symplectic blow-down of such a $G$-bundle along the two singular* (*simplectic*) *orbits of $G$.*

*Proof.* We consider three possible subcases: (IVa), (IVb) and (IVc).

(IVa) If $l \geq 2$ and $k \geq 2$, then $G_{\max} = S(U_{l+1} \times U_k \times U_1) \times G_0$, $G_{\min} = S(U_l \times U_1 \times U_{k+1}) \times G_0$, $G_{\text{reg}} = S(U_l \times U_k \times S^1) \times G_0$, and $Z(v) = S(U_l \times U_1 \times U_k \times U_1) \times G_0$. Here the inclusion $U_l \to U_{l+1}$ and $U_k \to U_{k+1}$ is canonical. Let $\mathcal{O} := G/(S(U_{l+1} \times U_{k+1}) \times G_0)$ be a coadjoint orbit of $G$. Let $\Pi_{\min}$ denote the natural $G$-equivariant projection from $G/G_{\min} \to \mathcal{O}$. In the same way we define the projection $\Pi_{\max}$. We observe that if the two points $m_{\max} \in G/G_{\max}$ and $m_{\min} \in G/G_{\min}$ are in the same gradient flow of the $G$-invariant function $H_G$ then their image under $\Pi_{\max}$ and $\Pi_{\min}$ coincide. Hence the projection $\Pi_{\min}$ and $\Pi_{\max}$ can be extended to a projection $\Pi : M \to \mathcal{O}$. Clearly the fiber is invariant under the $G$-action. The group $S(U_{l+1} \times U_{k+1})$ acts on the fiber of projection $\Pi$ from $M$

to $\mathcal{O}$ with three orbit types: the singular ones are $\mathbb{C}P^l$ and $\mathbb{C}P^k$ and the regular orbit is $S(U_{l+1} \times U_{k+1}))/S(U_l \times U_k \times S^1)$. Thus the fiber is diffeomorphic to $\mathbb{C}P^{l+k+1}$.

The simplest example of this case is $\mathbb{C}P^{l+k+1}$ with the standard action by $S(U_{l+1} \times U_{k+1}) \subset SU_{k+l+2}$.

(IVb) If $k = 1$, $l \geq 2$, then except the above decomposition for $G_{\max}$, $G_{\min}$, $G_{\text{reg}}$ and $Z(v)$ there is only the following possible subcase: $Z(v) = S(U_1 \times U_1 \times U_l) \times G_0$, $G_{\max} = S(U_2 \times U_l) \times G_0$, $G_{\min} = S(U_1 \times U_{l+1}) \times G_0$, and $G_{\text{reg}} = SU_l \times S^1 \times G_0$. Let $S^1_m$ be the subgroup of $Z(v)$ generated by the vector orthogonal to $Lie\, G_{\text{reg}}$ in $Lie\, Z(v)$. Denote by $\tilde{M}$ the suspension of $G/G_{\text{reg}}$. Clearly $\tilde{M}$ is diffeomorphic to $G \times_{Z(v)} S^2$, where $Z(v)$ acts on $S^2$ via the projection to $S^1_m$. According to Proposition 2.10 $\tilde{M}$ can be provided with a $G$-invariant symplectic form such that the reduced symplectic form at $G/Z(v)$ (considered at the "mean point" in $\tilde{M}$) is the same as that reduced from $M$. We claim that $M$ is a symplectic blow-down of $\tilde{M}$ along the two singular orbits $G/Z(v)_{\max}$ and $G/Z(v)_{\min}$. To see this we cut a $G$-invariant neighborhood of two $G$-singular orbits in $M$ (resp. $\tilde{M}$). By the very construction of $\tilde{M}$ these new symplectic manifolds are symplectomorphic. Hence follows the statement.

Now we shall show the existence of such a $G$-symplectic manifold. Denote by $k$ the Cartan subalgebra of $g$. By Kirwan's convexity theorem there are elements $v, \alpha \in k$ such that $Z(v) = S(U_1 \times U_1 \times U_l) \times G_0$, $Z(v + \alpha) = G_{\max}$, $Z(v - \alpha) = G_{\min}$. Duistermaat–Heckman tells us that the Chern class of the $S^1_m$-bundle $G/G_{\text{reg}} \to G/Z(v)$ is proportional to $\alpha$. Hence the Lie subalgebra $Lie\, G_{\text{reg}}$ is orthogonal to $\alpha$ in $Lie\, Z(v)$. We shall show that there are such elements $\alpha$ and $v$ satisfying the above condition.

Without lost of generality we assume that $G_0 = 1$. Thus $G = SU_{l+2}$. Write $v = (x_1, x_2, x_3, \ldots, x_3)$ ($l$ times) with $\sum x_i = 0$ and $x_1 \neq x_2$. Thus the equation for $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_3)$ is $\alpha_1 + \alpha_2 + l\alpha_3 = 0$, $x_2 + \alpha_2 = x_3 + \alpha_3$ (and is not zero), $x_1 - \alpha_1 = x_2 - \alpha_2$ (and is not zero). The solution to these equations is $(l+2)\alpha_1 = l(x_1 - x_2)$, $\alpha_2 = \alpha_1 - x_1 + x_2 = (l-1)x_1 - (2l-1)x_2$, $\alpha_3 = \alpha_2 + x_2 - x_3 = (l-1)(x_1 - 2x_2) - x_3$. The only thing need to check is the fact that $Z_{\max}/G_{\text{reg}} = S^{2l-1}$, $Z_{\min}/G_{\text{reg}} = S^{2k-1}$, where $G_{\text{reg}}$ is the subgroup generated by the subalgebra orthogonal to the vector $\alpha$. We can do it by finding an orthogonal representation of $G_{\min}$ (resp. $G_{\max}$) on $\mathbb{C}^2$ (resp. $\mathbb{C}^l$) such that it acts on $S^3$ (resp. $S^{2l-1}$) transitively with $G_{\text{reg}}$ as an isotropy group (see also [AA] which includes a corresponding Borel's table of the groups transitively acting on spheres).

With these data at hand it is easy to construct a $G$-invariant symplectic structure on the $G$-manifold $(G_{\min}, G_{\text{reg}}, G_{\max})$ by the same lifting construction as in the proof of Proposition 2.10. Namely we chose the family of symplectic form on $G/Z(v + t\alpha)$, $t \in [-1, 1]$, as the Kirillov–Kostant–Sourriau form.

(IVc) If $k = l = 1$, then except the decomposition analogous in subcase (IVb) (and hence subcase (IVa)) there is only the following possible cases with $Lie\, G_{\max} = Lie\, G_{\min} = su_2 \times Lie\, G_0$, $Lie\, Z_v = s(u_1 \times u_1) \times Lie\, G_0$. Using Kirwan's convexity theorem we conclude that this case never happens. □

Clearly Theorem 1.1 follows from Lemma 2.6, and Propositions 2.10, 2.12 and 2.14. We also note that any $M$ in cases (II)–(IV) can be considered as a symplectic blow-down of a

$G$-symplectic manifold in case I. To compute the cohomology ring of a symplectic blown we may use a method in [GH, Chap. 4, Section 6].

**Remark 2.15.** The case of a non-Hamiltonian symplectic action with cohomogeneity 1 of a compact Lie group $G$ is a bit more combinatorially complicated. The main observation in this case is the fact, analogous to Proposition 2.1, namely any principal orbit of such an action is an $S^1$-bundle over a homogeneous symplectic manifold (the Kirillov–Kostant–Sourriau theorem states that such a manifold is locally isomorphic to a coadjoint orbit of $G$ or an central extension of $G$). A simple case when the quotient space $Q = M/G$ is isomorphic to $S^1$ can be done easily because in this case, according to Alekseevskis' theorem [AA, Proposition 4.4], $M$ must be an extension of a primitive manifold $T^{l+1}$ with a free action of $\mathbb{Z}_k \times T^l$ by means of group $G$ and an epimorphism $\phi$ from a subgroup $H \subset G$ to $\mathbb{Z}_k \times T^l$, where $\mathbb{Z}_k \times T^l$ acts freely on $T^{l+1}$. Since $G/H$ is a principal orbit of $M$ it must be an $S^1$-bundle over a coadjoint orbit of a central extension of $G$.

## 3. Small quantum cohomology of some symplectic manifolds admitting a Hamiltonian action with cohomogeneity 1 of $U_n$

Small quantum cohomology [2] (or more precisely the quantum cup-product deformed at $H^2(M, \mathbb{C}) \subset H^*(M, \mathbb{C})$) was first suggested by Witten in context of quantum field theory and then has been defined mathematically rigorous for semi-positive (weakly monotone) symplectic manifolds by Ruan–Tian [RT] (see also [MS]) and recently for all compact symplectic manifolds by Fukaya and Ono [FO]. This quantum product structure is an important deformation invariant of symplectic manifolds (and recently Schwarz [Sch] has derived a symplectic fixed points estimate in terms of quantum cup-length). Nevertheless there are not so much examples of symplectic manifolds whose quantum cohomology can be computed (see [CF,FGP,GK,ST,RT,W]). The main difficulty in the computation of quantum cohomology is that if we want to compute geometrically it is not easy to "see" all the holomorphic spheres realizing some given homology class in $H_2(M, \mathbb{Z})$. (On the other hand, computational functorial relations for quantum cohomology are expected to be found.)

In this section we consider only the case of $M$ being a $\mathbb{C}P^k$-bundle over Grassmannian $Gr_k(N)$ of $k$-planes in $\mathbb{C}^N$: $M = U(N) \times_{(U(k)\times U(N-k),\phi)} \mathbb{C}P^k$, where $\phi$ acts on $\mathbb{C}P^k$ through the composition of the projection onto $U(k)$ with the embedding $U(k) \to U(k+1)$ and the standard action of $U(k+1)$ on $\mathbb{C}P^k$ ("standard" action means the projectivization of the standard linear action on $\mathbb{C}^{k+1}$). It is easy to see that the action on $\mathbb{C}P^k$ of the restriction of $\phi$ to $U(k)$ has two singular orbits: $\mathbb{C}P^{k-1}$ and a point, and its regular orbits are the sphere $S^{2k-1}$. According to the previous section we see that $M$ can be equipped with a $G$-invariant symplectic structure and a Hamiltonian action of $G = U(N)$ with the generic orbit of $G$-action on $M$ being isomorphic to $U(N)/(U(k-1) \times U(N-k))$ and

[2] For a definition and a formal construction of full quantum cohomology see [KM].

its image under the moment map $\mu: M \to u(n)$ is symplectomorphic to the flag manifold $U(N)/(U(1) \times U(k-1) \times U(N-k))$. With respect to Lemma 2.6 we see that $M$ belongs to case (I) if and only if $k = 1$, in this case $M$ is a toric manifold. We can also consider $M$ as the projectivization of the rank $(k+1)$ complex vector bundle over $Gr_k(N)$ which is the sum of the tautological $\mathbb{C}^k$-bundle $T_0$ and the trivial bundle $\mathbb{C}$. A special case of such $M$ is $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ whose quantum cohomology is computed in [RT, Example 8.6] (see also [KM]).

By Lemma 3.1 below $M$ admits a $G$-invariant monotone symplectic structure. To compute the small quantum cohomology algebra of $M$ we use several tricks well known before [ST,RT,W] (e.g. the use of Gromov's compactness theorem) and the positivity of intersection of complex submanifold. (In our monotone case we can also use the fact that the projection to the base $Gr_k(N)$ of a holomorphic sphere in $M$ is also a holomorphic sphere in $Gr_k(N)$ with area less than or equal to the area of the original sphere.) Thus we can solve this question in our cases positively. It seems that by the same way we can give a recursive rigorous computation of small quantum cohomology ring of full or partial flag varieties, since any $k$-flag manifold is a Grassmannian bundle over a $(k-1)$-flag manifold (see also [GK,CF,FGP] for other approachs to this problem).

Recall that [Bo] the cohomology algebra $H^*(Gr_k(N), \mathbb{C})$ is isomorphic to the factor-algebra of the algebra $\mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}[y_1, \dots, y_{N-k}]$ over the ideal generated by $S^+_{U(N)}$ $(x_1, \dots, y_{N-k})$ (see also Proposition A.4 in Appendix A). Geometrically $x_i$ is $i$th Chern class of the dual bundle of the tautological $\mathbb{C}^k$-vector bundle over $Gr_k(N)$, and $y_i$ is $i$th Chern class of the dual bundle of the other complementary $\mathbb{C}^{N-k}$-vector bundle over $Gr_k(N)$. Another description of $H^*(Gr_k(N), \mathbb{R})$ uses Schubert cells which form an additive basis, the Schubert classes, in $H^*(Gr_k(N), \mathbb{R})$ (see e.g. [FGP] and the references therein for the relation between two approaches). Summarizing we have (see e.g [ST,MS])

$$H^*(Gr_k(N), \mathbb{C}) = \frac{\mathbb{C}[x_1, \dots, x_k]}{\langle y_{N-k+1}, \dots, y_N \rangle}$$

where $y_{N-k+j} := -\sum_{i=0}^{N-k+j} x_i y_{N-k+j-i}$ (are defined inductively). The first Chern class of $T_* Gr_k(N)$ is $N x_1$.

The quantum cohomology of $Gr_k(N)$ was computed in [ST,W]. Now let us compute the quantum cohomology algebra $QH^*(M, \mathbb{C})$. Denote by $f$ the Poincare dual to the big singular orbit $U(N)/(U(1) \times U(k-1) \times U(N-k))$ in $M$. Let $x_1, \dots, x_k$ be the generators of $H^*(Gr_k(N), \mathbb{C})$ as above. It is easy to see that the first Chern class of $T_* M$ is $(N-1)x_1 + (k+1)f$. Then the minimal Chern number of $T_* M$ is GCD $(N-1, k+1)$ (because the $H_2(M, \mathbb{Z})$ is generated by $H_2(Gr_k(N))$ and $H_2(\mathbb{C}P^k)$).

**Lemma 3.1.**

(i) *We have*

$$H^*(M, \mathbb{C}) = \frac{\mathbb{C}[f, x_1, \dots, x_k]}{\langle f(f^k - x_1 f^{k-1} + \cdots + (-1)^k x_k), y_{N-k+1}, \dots, y_N \rangle}.$$

(ii) *$M$ admits a $G$-invariant monotone symplectic structure.*

(i) The formula is known in more general context [BT, Chap. 4, Section 20; GH, Chap. 4, Section 6]. But in our simple case we shall supply here a simple proof. To derive Lemma 3.1 from the proof of Proposition 2.12 it suffices to show that

$$PD_M(Gr_k(N)) = f^k - x_1 f^{k-1} + \cdots + (-1)^k x_k. \tag{3.1}$$

To prove (3.1) we denote $PD_M(Gr_k(N))$ by a polynomial $P_k(f, x_1, \ldots, x_k)$. By considering the restriction of $PD_M(Gr_k(N))$ to the small orbit $Gr_k(N)$ we conclude that the lowest term (free of $f$) of $P_k$ is $(-1)^k x_k$. To define the other terms of $P_k$ we consider the restriction of $PD_M(Gr_k(N)) = P_k$ to the submanifold $\bar{M} \subset M$, which is the $\mathbb{C}P^k$ bundle over $Gr_{k-1}(N-1)$. Let $M'$ be a submanifold of $\bar{M}$ which is defined as $M$ but over $Gr_{k-1}(N-1)$. Using the formula

$$(P_k)_{|\bar{M}} = PD_{\bar{M}}(Gr_{k-1}(N-1)) = PD_{\bar{M}}(M') \cdot PD_{M'}Gr_{k-1}(N-1)$$

and the fact that$PD_{\bar{M}}(M') = f$, we conclude (by using the induction step) that $P_k$ equals RHS of (3.1).

(ii) It is well known that $Nx_1$ is a symplectic class in $H^2(Gr_k(N), \mathbb{R})$. By checking the non-degeneracy of the family of $U(N)$-invariant forms $(Nx_1 + t(k+1)f)$ at a point $T_e((U(N))/U(1) \times U(k-1) \times U(N-k)$ we conclude that the condition for the existence of an invariant symplectic form in the proof of Proposition 2.12 holds. Hence $M$ admits a $G$-invariant monotone symplectic structure.

According to a general principle for computing the small quantum cohomology ring of a monotone symplectic manifold $(M, \omega)$ we need to compute only the quantum relations [ST,W]. More precisely, let $g_i(z_1, \ldots, z_m)$ be polynomials generating the relations ideal of the cohomology algebra $H^*(M, \mathbb{C})$ generated by $\{z_i\}$. Then $z_i$ are also generators of the small quantum algebra $QH^*(M, \mathbb{C}) = H^*(M, \mathbb{C}) \otimes \mathbb{Z}[q]$ with the new relations $\hat{g}_i(z_i) = qP_i(z_i, q)$. Here $q$ is the quantum variable, $\hat{g}_i$ is the polynomial defined by $g_i$ with respect to the quantum product in $QH^*(M, \mathbb{C})$. Denote the quantum product by $\star$. There are several equivalent approachs to small quantum cohomology but we use notations (and formalism) in [MS].

**Theorem 3.2.** *Let $M$ satisfy the condition $2(k+1) = N-1$ and as before, let $P_k$ denote the Poincare dual to $Gr_k(N)$. Then its small quantum cohomology ring is isomorphic to*

$$QH^*(M) = \frac{\mathbb{C}[f, x_1, \ldots, x_k, q]}{\langle f \star P_k = q, y_{N-k+1}, \ldots, y_{N-1}, y_N = (-1)^{k+1} q^2 f\rangle}.$$

*Proof.* Recall that (see e.g. [McDS]) the moduli space $\mathcal{M}_A(M)$ of holomorphic spheres realizing class $A \in H_2(M, \mathbb{Z})$ gives a non-trivial contribution the quantum product of $a \star b$, $a, b \in H^*(M, \mathbb{C})$, if there is an element $c \in H^*(M, \mathbb{C})$ such that the Gromov–Witten invariant $\Phi_A(PD(a), PD(b), PD(c)) \neq 0$. In this case we have

$$\deg(a) + \deg(b) \leq \dim M + 2c_1(A) \leq \deg a + \deg b + \dim M, \tag{3.2}$$

which is also called a degree (dimension) condition.

Recall that in our case the minimal Chern number of $M$ is $(k+1)$. Thus from (3.2), Lemma 3.1 and the monotonicity condition we see immediately that if the moduli space $\mathcal{M}_A(M)$ has a non-trivial contribution to the quantum relation then $0 < c_1(A) \le 2(k+1)$. Hence $A$ must be one of the five following homology classes.

(C1) the homology class $[u]$ generating the homology group $H_2(\mathbb{C}P^k, \mathbb{Z}) = \mathbb{Z}$ of the fiber $\mathbb{C}P^k$;

(C2) class $2[u]$;

(C3) class $[v]$ which can be realized as a holomorphic sphere on one singular orbit $G(m_+)$ which is diffeomorphic to $Gr_k(N)$ (see also the previous section);

(C4) the (exceptional) class $[v]-[u]$.

(C5) the (double exceptional) class $2([v]-[u])$.

Note that $[u]$ and $[v]$ are the generators of $H_2(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

Let us consider the moduli space of holomorphic spheres in class $[u]$. It is easy to see that with respect to the standard integrable complex structure $J$ on $M$ the $J$-holomorphic spheres realizing this class $[u]$ are exactly the complex lines of the fiber $\mathbb{C}P^k$. The simplest way to see this is to look at the projection of these holomorphic spheres on the base $Gr_k(N)$. (It may be possible to see this by using the curvature estimate in [L]. This curvature estimate could be able to show that the minimal sectional curvature distribution in $M$ consists of 2-planes in the tangent space of the fiber $\mathbb{C}P^k$. Using the same curvature estimate we have characterized the space of holomorphic spheres of minimal degree in complex Grassmannian and other complex symmetric spaces [L] as the space of Helgason spheres.) A simple computation shows that the virtual dimension of the moduli space $\mathcal{M}_u(\mathbb{C}P^1, M)$ of $J$-holomorphic spheres realizing $[u]$ equals the real dimension of this space and equals $2(k+1) + 2k + 2N(N-k)$. We can also apply the regularity criterion $H^1(\mathbb{C}P^1, f^*(T_*M)) = H^1(\mathbb{C}P^1, \bar{f}^*(T_*(\mathbb{C}P^k)) = 0$. Here $f$ is a $J$-holomorphic map $\mathbb{C}P^1 \to M$ and $\bar{f}$ is its restriction on the fiber $\mathbb{C}P^k$.

Now let us compute the contribution of the moduli space $\mathcal{M}_{[u]}(M)$ to the quantum relations. First we note that by dimension reason the quantum polynomial of degree less than $(k+1)$ must coincide with the usual polynomial (in the ring $H^*(M, \mathbb{C})$). Thus to compute the contribution of $\mathcal{M}_{[u]}(M)$ to the first defining relation it suffices to compute the following Gromov–Witten invariants with $1 \le l \le k+1$

$$\Phi_{[u]}(PD(f^l), PD(x_{k+1-l}), pt), \tag{3.3a}$$

$$\Phi_{[u]}(PD(f^k, PD(f), pt). \tag{3.3b}$$

We claim that the Gromov–Witten invariant in (3.3a) equals zero. We observe that $PD_M(x_{k+1-l}) = j^{-1}(PD_B(x_{k+1-l}))$, where as in the previous section we denote by $j$ the projection of $M$ to $Gr_k(N)$. Hence, taking into account that $[u]$ is a "fiber" class we see immediately, by dimension reason, that there is no holomorphic curve in class $[u]$ which intersects $j^{-1}(PD_B(x_{k+1-l}))$ and goes through a $PD(f^l)$.

We claim that the Gromov–Witten invariant in (3.3b) equals 1. To prove this we fix a fiber $\mathbb{C}P^k$ which contains the given point $pt$. We observe that the singular orbit representing $PD_M(f)$ intersects with each fiber $\mathbb{C}P^k$ at a divisor $\mathbb{C}P^{k-1}$. Finally we note that $PD_M(f^k)$

intersects with the fixed $\mathbb{C}P^k$ at one point because $f^k([\mathbb{C}P^{k-1}]) = 1$. Since there is exactly one complex line through the given two points in $\mathbb{C}P^n$ (and this line always intersects the divisor $\mathbb{C}P^{k-1} \subset \mathbb{C}P^k$) we deduce that the Gromov–Witten invariant in (3.3b) is 1.

Summarizing we get

$$f \star_{[u]} P_k = q \tag{3.3c}$$

(here the LHS of (3.3c) denotes the quantum polynomial, deformed by $[u]$).

Next we shall compute the contribution of $\mathcal{M}_{[u]}$ to the "old" defining relation $y_j$, $j = N-k+1, N$. First we shall show that

$$\Phi_{[u]}(PD_M(x_p), PD_M(y_{j-p}), PD_M[w]) = 0 \tag{3.3d}$$

for any $[w] \in H^*(M)$ with degree equal $\dim M + 2(k+1) - 2j$. Using the formula $PD_M[j^*(y)] = j^{-1}PD_B[y]$ for the Poincare dual of a pull-back cohomology class of the base of a fiber bundle we observe that if (3.1) is not zero then $PD_M[w]) \cap PD_M(x_p) \cap PD_M(y_{j-p}) \neq \emptyset$. But it is impossible by the dimension reason.

Thus there remain possibly four other non-trivial contributions to the quantum relations. The first one is related to the Gromov–Witten invariants

$$\Phi_{[2u]}(PD_M(x_p), PD_M(y_{j-p}), PD_M(w)), \tag{3.4}$$

the second to the Gromov–Witten invariants

$$\Phi_{[v]}(PD_M(x_p), PD_M(y_{j-p}), PD_M(w)), \tag{3.5}$$

and the two other Gromov–Witten invariants related to the (exceptional) classes $[v]-[u]$ and $2([v]-[u])$.

Here in the cases (C2) and (C3) the degree of $w$ must be $\dim M + 4(k+1) - 2j$.

To compute (C2) we use a generic almost complex structure $J_{\mathrm{reg}}$ nearby the integrable one. Thus the image of $J_{\mathrm{reg}}$-holomorphic spheres in class $2[u]$ must in a (arbitrary) small neighborhood of a complex line in the fiber $\mathbb{C}P^k$, that is the projection of a $J_{\mathrm{reg}}$-holomorphic sphere in class $[u]$ must be in a ball of radius $\varepsilon/2$. Now we can use the same argument as before. Since $PD_M(x_p) \cap PD_M(y_{j-p}) \cap PD_M(w) = \emptyset$ there exists a positive number $\varepsilon$ such that the $\varepsilon$-neighborhood of these cycles also do not have a common point. Now looking at the projection of these cycles on the base $Gr_k(N)$ we conclude that the contribution (C2) is zero.

In order to compute the contribution (C3) we need to know the moduli space of the holomorphic spheres in class $[v]$ whose dimension is $\dim M + 4(k+1) = \dim Gr_k(N) + 6k + 4 = \dim Gr_k(N) + 2N + 2(k-1)$. We pick up the standard integrable complex structure. We claim that all these holomorphic spheres can be realized as holomorphic sections of $\mathbb{C}P^k$-bundle over $\mathbb{C}P^1_{[v]}$, where $\mathbb{C}P^1_{[v]}$ is a holomorphic sphere of minimal degree in $Gr_k(N)$. Indeed over this $\mathbb{C}P^1$ the bundle $\mathbb{C}P^k$ is the projectivization of the sum of $(k+1)$ holomorphic line bundles with $k$ Chern numbers being 0 and one number being $(-1)$. Thus for any holomorphic sphere $(S^2, f)$ which is a holomorphic section of the $\mathbb{C}P^k$ bundle over $\mathbb{C}P^1$ we have $H^1(S^2, f^*(T_*M)) = H^1(S^2, f^*(T_*\mathbb{C}P^k)) = 0$. To show that these holomorphic sections exhaust all the holomorphic spheres in the class $[v]$ we look at their projection on the base $Gr_k(N)$.

Now let us to compute (C3) with $j = N - 1$ or $j = N$ (by dimension condition (3.2) those are the only cases which may enter into the quantum relations).

If $j = N - 1$ then the contribution in (C3) must be 0 since we know that on the base $B = Gr_k(N)$ there is no holomorphic curve of minimal degree which go through the cycle $PD_B(x_p)$ and $PD_B(y_{N-p-1})$ (by dimension reason).

If $j = N$ then there are two possibilities for $PD_M(w)$, namely they are $[u]$ and $[v]$ – the generators of $H^2(M, \mathbb{Z})$.

Let us consider the first case, i.e. $PD_M(w)$ is a holomorphic sphere $u$ in the fiber $\mathbb{C}P^k$. The induction argument on $Gr_k(N)$ [ST,W] shows that $p$ in (C3) must be $k$ and there is a unique (up to projection $j$) holomorphic sphere in class $[v]$ which intersects with $PD_M(x_k)$ and $PD_M(y_{N-k})$ and satisfies the following property: Its image under the projection $j$ goes through the fixed point $j(u) \in Gr_k(N)$. Hence we can reduce our computation of the corresponding contribution in (C3) to the related Gromov–Witten invariant in the $\mathbb{C}P^k$-bundle over $\mathbb{C}P^1_{[v]}$. Thus we get

$$\Phi_{[v]}(PD_M(x_k), PD_M(y_{N-k}), [u]) = (-1)^{k+1}. \tag{3.6a}$$

Now let us consider the second case, i.e. $PD_M(w)$ is the class $[v]$ realized by a holomorphic section of the $\mathbb{C}P^k$-bundle over the $\mathbb{C}P^1$. Clearly there is only one holomorphic section passing through a given point in this bundle. Thus we get

$$\Phi_{[v]}(PD_M(x_k), PD_M(y_{N-k}), [v]) = (-1)^{k+1}. \tag{3.6b}$$

In order to compute (C4) let us consider the moduli space of holomorphic spheres in the class $[v] - [u]$. We have two arguments to show that there is no $J$-holomorphic sphere in this class. The simplest argument was suggested by Kaoru Ono. Namely considering the intersection of a holomorphic sphere in this class with the big singular orbit $U(N)/(U(1) \times U(k-1) \times U(N-k))$ yields that there is no holomorphic sphere in this class. The another (longer) argument uses the area comparison. Clearly the area of such a holomorphic sphere equals the value $\omega([v] - [u])$. On the other hand the projection to $Gr_k(N)$ of a holomorphic sphere in this class has area $\omega([v]) > \omega([v] - [u])$. (The projection decreases the area because of Duistermaat–Heckman theorem applied to our monotone case.) Thus there is no $J$-holomorphic sphere in this class. Since the class $[u] - [v]$ is indecomposable in the Gromov sense it follows from the Gromov compactness theorem that for nearby generic almost complex structure $J'_{\mathrm{reg}}$ there is also no $J'_{\mathrm{reg}}$-holomorphic sphere. Thus there is no quantum contribution of this class.

Finally to compute (C5) we consider the quantum contribution associated to the class $2([v] - [u])$. The space of $J$-holomorphic spheres in this class is empty by the same reason as above (two arguments). Finally by using the Gromov compactness theorem we can show the existence of a regular almost complex structure $J_{\mathrm{reg}}$ nearby $J$ such that there is no $J_{\mathrm{reg}}$-holomorphic sphere in this class. (Because if bubbling happens, they must be holomorphic spheres in class $[v] - [u]$, which is also impossible.)

Summarizing we get that the only new quantum relations are those involving (3.3e), (3.5a) and (3.5b). Note that $f$ is defined uniquely by the condition $f(u) = 1 = f(v)$. This completes the proof of Theorem 3.2. □

**Remark 3.3.** Since the rank of $H_2(M)$ is 2 it is more convenient to take two quantum variables $q_1, q_2$. In this case our computations give a (slightly) formal different answer, namely $(R2) = q_1$ and $y_N = (-1)^{k+1}(q_1^2 f_1 + q_2^2 f_2)$. Here $f_1$ and $f_2$ form a basis of $Hom(H_2(M, \mathbb{C}), \mathbb{C}) = H^2(M, \mathbb{C})$ which is dual to the basis $([u], [v]) \in H_2(M, \mathbb{C})$.

**Remark 3.4.** Let $M$ be a symplectic manifold as in Theorem 3.2.

(i) It follows immediately from Theorem 3.2 and Schwarz's result [Sch] that the any exact symplectomorphism on $M$ has at least $k+1$ fixed points.

(ii) It seems that after a little work we can apply the result in [HV] to show that the Weinstein conjecture also holds for those $M$.

## 4. Compact symplectic manifolds admitting symplectic action of cohomogeneity 2

A direct product of $(M_1, \omega_1)$ and $(M_2, \omega_2)$ is a symplectic manifold which admits a symplectic action of cohomogeneity 2 provided that either both $(M_i, \omega_i)$ admit symplectic action of cohomogeneity 1 or $(M_1, \omega_1)$ is a homogeneous symplectic manifold and $(M_2, \omega_2)$ has dimension 2. These examples are extremally opposite in a sense that, in the first case the normal bundle of any regular orbit is isotropic, and in the second case the normal bundle is symplectic.

**Proposition 4.1.** *Suppose that an action of $G$ on $(M^{2n}, \omega)$ is Hamiltonian and* $\dim M/G = 2$. *Then either all the principal orbits of $G$ are symplectic (simultaneously), or all the principal orbits of $G$ are coisotropic (simultaneously). In the first case a principal orbit is isomorphic to a coadjoint orbit of $G$, in the last case a principal orbit must be a $T^2$-bundle over a coadjoint orbit of $G$.*

*Proof.* Since the set $M^{\mathrm{reg}}$ of regular points in $M^{2n}$ is open and dense in $M^{2n}$, and the property of being symplectic is an open condition, it suffices to show that there is an open, dense, $G$-invariant set $M \subset M^{\mathrm{reg}}$ such that all the orbit $G(x) \subset M$ is symplectic (or coisotropic simultaneously). We consider the moment map $\mu : M^{2n} \to g^* = g$. By Sard's theorem the set $S_\mu$ of points $x$ in $M^{2n}$, where the dimension $d$ of $\mu^{-1}\{\mu(x)\}$ is maximal, is open and dense in $M^{2n}$. Let $M_\mu^{\mathrm{reg}}$ be the set in $M$ consists of points $x$ such that $\mu(G(x))$ is a orbit of maximal dimension in $\mu(M)$. Using Kirwan's theorem we see that $M_\mu^{\mathrm{reg}}$ is an open and dense set in $M$. We claim that we can take $M$ as the intersection of $S_\mu$ with $M_\mu^{\mathrm{reg}}$ and the set of regular points in $M^{2n}$. Using formula (2.3) we note that $d \le 2$. Since the dimensions of $G(x)$ and of $\mu(G(x))$ are even if $x \in M$, we get that $d$ must be either 0 or 2. First we suppose that $d = 0$. Since $G$ is connected all the other principal orbit $G(m')$ in $M$ also connected, and since $\mu(G(m))$ is simply connected, all the principal orbits in $M$ must be diffeomorphic to $\mu(G(m))$ (and hence are symplectic). Clearly if orbit is symplectic then the restriction of $G$-action on it is also Hamiltonian, thus by Kirillov–Kostant–Sourriau theorem, it must be isomorphic to a coadjoint orbit of $G$. Now let us assume that the "generic" dimension $d$ of $\mu^{-1}\{\mu(m)\}$ is 2. Since the dimension of $\mu(G(x))$ is a constant for $x \in M$, we conclude that either all $G(x)$, for $x \in M$ is either symplectic simultaneously or isotropic simultaneously. In the last case $\mu^{-1}\{\mu(x)\} \subset G(x)$ and $\mu(G(x)) = G(x)/\mu^{-1}\{\mu(x)\}$. Arguing as in the proof of Proposition 2.1 we see that $\mu^{-1}\{\mu(x)\}$ admits a nowhere zero vector fields $sgrad\,\mathcal{F}_{v_1}$ and $sgrad\,\mathcal{F}_{v_2}$. Thus it must be an isotropic torus. □

**Remark 4.2.**
(i) The quotient space $\mu(M)/G$ is either a point or a convex two-dimensional polytop.
(ii) If the action of $G$ is Hamiltonian and the principal orbit is symplectic then the condition that $\mu(M)/G$ is a point is equivalent to the fact that $d$ (in the proof of Proposition 4.1) equals 2. In this case $M$ is diffeomorphic to a bundle over a coadjoint orbit of $G$ whose fiber is a two-dimensional surface.

The first statement in Remark 4.2 follows from the proof of Proposition 4.1 and Kirwan's theorem on convexity of moment map. The second statement follows by considering the moment map.

**Proposition 4.3.** *Suppose that the action of $G$ is Hamiltonian, the number $d$ (in the proof of Proposition 4.1) is zero and the action of $G$ on $\mu(M)$ has only one orbit type. Then $M$ is $G$-diffeomorphic to a fiber bundle over a two-dimensional surface $\Sigma$, whose fiber is isomorphic to a coadjoint orbit of $G$.*

Indeed, by the dimension reason in this case there is also only one orbit type of $G$-action on $M$. Note that such a bundle always admits a $G$-invariant symplectic structure.

If the principal orbits of $G$ in $M$ are coisotropic then $P = \mu(M)/G$ is always a two-dimensional convex polytop.

**Proposition 4.4.** *If the action of $G$ on $M$ is Hamiltonian and the principal orbit of $G$ is coisotropic then $M$ is diffeomorphic to the bundle of ruled surface over a coadjoint orbit of $G$ provided that the action of $G$ on $\mu(M)$ has only one orbit type.*

*Proof.* In this case $M$ admits a projection $\pi$ over a coadjoint orbit $\mu(G(m))$ with fiber $\pi^{-1}$ being a symplectic 4-manifold. This symplectic 4-manifold admits a $T^2$-Hamiltonian action. Hence it must be a rational or ruled surface (see [Au]). □

## Appendix A. Homogeneous symplectic spaces of compact Lie groups

First we recall a theorem of Kirillov–Kostant–Sourriau (see e.g. [Kir]).

**Theorem A.1.** *A symplectic manifold admitting a Hamiltonian homogeneous action of a connected Lie group $G$ is isomorphic to a covering of a coadjoint orbit of $G$.*

If $G$ is a connected compact Lie group then all its coadjoint orbits are simply connected. Thus in this case we have the following simple result.

**Corollary A.2.** *A symplectic manifold admitting a Hamiltonian homogeneous action of a connected compact Lie group $G$ is a coadjoint orbit of $G$.*

*Table A.3.* We present here a list of all coadjoint orbits of simple compact Lie groups. Recall that a coadjoint orbit through $v \in g$ can be identified with the homogeneous space $G/Z(v)$ with $Z(v)$ being the centralizer of $v$ in $G$. Element $v$ in a Cartan algebra $Lie\, T^k \subset g$ is regular iff for all root $\alpha$ of $g$ we have $\alpha(v) \neq 0$. In this case $Z(v)$ is the maximal torus $T^k$ of $G$. If $v$ is a singular element with $\alpha_i(v) = 0$ then $Lie\, Z(v)$ is a direct sum of the subalgebra in $g$ generated by the roots $\alpha_i$ and $Lie\, T^k$. To identify the type of this subalgebra $Lie\, Z(v)$ we observe that $Lie\, T^k$ is its Cartan subalgebra and the root system of $Lie\, Z(v)$ consists of those roots $\alpha$ of $G$ such that $\alpha(v) = 0$. Looking at tables of roots of simple Lie algebras [OV] and their Dynkin schemes we get easily the following list (which perhaps could be found somewhere else)

(A) If $G = SU_{n+1}$ then $Z(v) = S(U_{n_i} \times \cdots \times U_{n_k})$, $\sum n_i = n + 1$.

(B), (C), (D) If $G$ is in $B_n$, $Z_n$ or $D_n$ then $Z(v)$ is a direct product $U_{n_1} \times \cdots \times U_{n_k} \times G_p$ with $rkG_p + \sum n_i = rkG$, and $G_p$ and $G$ must be from the same series (B), (C), (D).

Analogously but more combinatorically complicated are the types of $Z(v)$ in the exceptional series. Note that all the listed below simple exceptional groups are simply connected.

($E_6$) Except the regular orbits with $Z(v) = T^6$ we also have other possible singular orbits with $Z(v) = S(U_{k_1} \times \cdots \times U_{k_n})$ with $n \geq 2$, $\sum k_i = 7$ and $T^k \times Spin_{6-k}$ with $k = 1, 2$.

($E_7$) Analogously. Possible are also $Z(v) = T^1 \times SU_2 \times Spin_{10}$ and $Z(v) = T^1 \times E_6$.

($E_8$) Analogously. Possible are also $T^1 \times E_7$ and $T^1 \times SU_2 \times E_6$.

($F_4$) Singular orbits can have $Z(v)$ being $T^2 \times SU_3$, $T^2 \times SU_2 \times SU_2$ or $T^1 \times Spin_7$ and $T^1 \times Sp_3$.

($G_2$) Except the regular orbit $G_2/T^2$ there are also singular orbit $G_2/SU_2 \times T^1$.

To compute the cohomology ring of $G/Z(v)$ we use:

**Proposition A.4** [Bo, Theorem 26.1].

(i) *The cohomology algebra $H(G/Z(v), \mathbb{R})$ is a factor-algebra $S_{Z(v)}$ over the ideal generated by $\rho_R^*(S_G^+)$ which equals the characteristic subalgebra.*

(ii) *Let $s_1 - 1, \ldots, s_l - 1$ and correspondingly, $r_1 - 1, \ldots, r_l - 1$ be degree of the generators in $H^*(G)$ and $H^*(Z(v))$. Then the Poincare polynomial of $G/Z(v)$ equals*

$$\frac{(1 - t^{s_1}) \cdots (1 - t^{s_l})}{(1 - t^{r_1}) \cdots (1 - t^{r_l})}.$$

Here $S_G$ is the algebra of $G$-invariant polynomials in $g$ and $S_G^+$ is its subalgebra which is generated by monomials of positive degree.

**Remark A.5.** All the $G$-invariant symplectic form on $G/Z(v)$ are compatible with the (obvious) $G$-invariant complex structure. Thus all of them are deformation equivalent to a monotone symplectic form.

**Remark A.6.** For any symplectic form $\omega$ on a homogeneous space $M^{2n}$ of a compact Lie group $G$ the averaged form $\omega^G$ is a $G$-invariant symplectic form in the same cohomology class $[\omega]$. Thus the necessary and sufficient condition for the existence of a symplectic form in a cohomology class $[\omega] \in H^2(M^{2n}, \mathbb{R})$ is that $[\omega]^n > 0$. As another consequence we see that any homogeneous space of a compact Lie group which admits a symplectic structure is diffeomorphic to a homogeneous symplectic manifold. But it is not true for a compact manifold of cohomogeneity 1 (or higher cohomogeneity). For example $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admits a $SU(3)$ action of cohomogeneity 1 (with no fixed point) but no symplectic form invariant under this action.

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